Uitwerking FINAL EXAM

Kwantumfysica 1 - 1 NOV. 2012

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Problem 1

a) The Hamiltonian is only a function of $\hat{S}_z$, so $\hat{H}$ and $\hat{S}_z$ commute ($[\hat{H}, \hat{S}_z] = 0$).

$\hat{S}_x$ and $\hat{S}_y$ do not commute with the Hamiltonian because $[\hat{S}_x, \hat{S}_z] \neq 0$ and $[\hat{S}_y, \hat{S}_z] \neq 0$.

b) The Hamiltonian has the same eigenstates as $\hat{S}_z$, so these are simply

$$\begin{cases} |\uparrow\rangle & \text{(spin-up along the z-axis)} \\ |\downarrow\rangle & \text{(spin-down along the z-axis)} \end{cases}$$

The energy eigen value for the state $|\uparrow\rangle$ is

$$\hat{H}|\uparrow\rangle = \underbrace{-\gamma B_z \cdot +\tfrac{1}{2}\hbar}_{\text{energy eigen value for } |\uparrow\rangle} |\uparrow\rangle$$, where we use $\hat{S}_z|\uparrow\rangle = \frac{1}{2}\hbar|\uparrow\rangle$

$\Rightarrow \boxed{E_\uparrow = -\frac{1}{2}\gamma\hbar B_z}$

The energy eigen value for the state $|\downarrow\rangle$ is

$$H|\downarrow\rangle = \underbrace{-\gamma B_z \cdot -\tfrac{1}{2}\hbar}\, |\downarrow\rangle$$, where we used $\hat{S}_z|\downarrow\rangle = -\frac{1}{2}\hbar|\downarrow\rangle$

$\Rightarrow$ the energy eigen value for $|\downarrow\rangle$ is

$$\boxed{E_\downarrow = +\tfrac{1}{2}\gamma\hbar B_z}$$

$E_\uparrow < E_\downarrow$, so $|\uparrow\rangle$ is the ground state and $|\downarrow\rangle$ the only excited state.

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c) Resonant driving of quantum dynamics requires

$$hf = E_\downarrow - E_\uparrow = \gamma\hbar B_z \Rightarrow \boxed{f = \frac{\gamma B_z}{2\pi}} \Rightarrow$$

$$f = 27.11 \text{ MHz}$$

d) $\boxed{\hat{S}_z}$

$$\langle\uparrow|\hat{S}_z|\uparrow\rangle = +\tfrac{1}{2}\hbar$$
$$\langle\downarrow|\hat{S}_z|\downarrow\rangle = -\tfrac{1}{2}\hbar$$
$$\langle\uparrow|\hat{S}_z|\downarrow\rangle = 0$$
$$\langle\downarrow|\hat{S}_z|\uparrow\rangle = 0$$

Further use the matrix representation in S_z basis

$$|\uparrow\rangle \Leftrightarrow \begin{pmatrix}1\\0\end{pmatrix}, \quad |\downarrow\rangle \Leftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$$

$$\hat{S}_x \Leftrightarrow \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad \hat{S}_y \Leftrightarrow \frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$$

$\boxed{S_x}$

$$\langle\uparrow|\hat{S}_x|\uparrow\rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0$$
$$\langle\downarrow|\hat{S}_x|\downarrow\rangle = \frac{\hbar}{2}(0\ 1)\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 0$$
$$\langle\uparrow|\hat{S}_x|\downarrow\rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = +\frac{\hbar}{2}$$
$$\langle\downarrow|\hat{S}_x|\uparrow\rangle = \frac{\hbar}{2}(0\ 1)\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = +\frac{\hbar}{2}$$

$\boxed{S_y}$

$$\langle\uparrow|\hat{S}_y|\uparrow\rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0$$
$$\langle\downarrow|\hat{S}_y|\downarrow\rangle = \frac{\hbar}{2}(0\ 1)\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 0$$
$$\langle\uparrow|\hat{S}_y|\downarrow\rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -i\frac{\hbar}{2}$$
$$\langle\downarrow|\hat{S}_y|\uparrow\rangle = \frac{\hbar}{2}(0\ 1)\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = +i\frac{\hbar}{2}$$

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e) Define $\omega_\uparrow = \frac{E_\uparrow}{\hbar}$ and $\omega_\downarrow = \frac{E_\downarrow}{\hbar}$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi_0\rangle = \sqrt{\tfrac{2}{3}}e^{-i\omega_\uparrow t}|\uparrow\rangle + i\sqrt{\tfrac{1}{3}}e^{-i\omega_\downarrow t}|\downarrow\rangle$$

$$\langle\hat{S}_z\rangle(t) = \langle\psi(t)|\hat{S}_z|\psi(t)\rangle$$

$$= \left(\sqrt{\tfrac{2}{3}}e^{+i\omega_\uparrow t}\langle\uparrow| - i\sqrt{\tfrac{1}{3}}e^{+i\omega_\downarrow t}\langle\downarrow|\right)\hat{S}_z\left(\sqrt{\tfrac{2}{3}}e^{-i\omega_\uparrow t}|\uparrow\rangle + i\sqrt{\tfrac{1}{3}}e^{-i\omega_\downarrow t}|\downarrow\rangle\right)$$

$$= \tfrac{2}{3}\langle\uparrow|S_z|\uparrow\rangle + \tfrac{1}{3}\langle\downarrow|\hat{S}_z|\downarrow\rangle$$

$$= +\tfrac{2}{3}\cdot\tfrac{1}{2}\hbar - \tfrac{1}{3}\cdot\tfrac{1}{2}\hbar = \tfrac{1}{6}\hbar$$

$$\langle\hat{S}_x\rangle(t) = \langle\psi(t)|\hat{S}_x|\psi(t)\rangle$$

$$= \left(\sqrt{\tfrac{2}{3}}e^{+i\omega_\uparrow t}\langle\uparrow| - i\sqrt{\tfrac{1}{3}}e^{+i\omega_\downarrow t}\langle\downarrow|\right)\hat{S}_x\left(\sqrt{\tfrac{2}{3}}e^{-i\omega_\uparrow t}|\uparrow\rangle + i\sqrt{\tfrac{1}{3}}e^{-i\omega_\downarrow t}|\downarrow\rangle\right)$$

$$= \tfrac{\sqrt{2}}{3}e^{-i(\omega_\downarrow-\omega_\uparrow)t}\cdot i\,\langle\uparrow|\hat{S}_x|\downarrow\rangle - \tfrac{\sqrt{2}}{3}e^{+i(\omega_\downarrow-\omega_\uparrow)t}\cdot i\langle\downarrow|\hat{S}_x|\uparrow\rangle$$

$$= \tfrac{\sqrt{2}}{3}i\tfrac{\hbar}{2}\Big(\underbrace{e^{-i(\omega_\downarrow-\omega_\uparrow)t} - e^{+i(\omega_\downarrow-\omega_\uparrow)t}}_{(\cancel{\cos} - i\sin) - (\cancel{\cos} + i\sin)}\Big)$$

$$= \tfrac{\sqrt{2}\hbar}{6}\,i\left(-i\sin(\omega_\downarrow-\omega_\uparrow)t - i\sin(\omega_\downarrow-\omega_\uparrow)t\right)$$

$$= \tfrac{\sqrt{2}}{3}\hbar\,\sin(\omega_\downarrow-\omega_\uparrow)t$$

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$$\langle S_y\rangle(t) = \langle\psi(t)|\hat{S}_y|\psi(t)\rangle$$

$$= \left(\sqrt{\tfrac{2}{3}}e^{+i\omega_\uparrow t}\langle\uparrow| - i\sqrt{\tfrac{1}{3}}e^{+i\omega_\downarrow t}\langle\downarrow|\right)\hat{S}_y\left(\sqrt{\tfrac{2}{3}}e^{-i\omega_\uparrow t}|\uparrow\rangle + i\sqrt{\tfrac{1}{3}}e^{-i\omega_\downarrow t}|\downarrow\rangle\right)$$

$$= \tfrac{\sqrt{2}}{3}e^{-i(\omega_\downarrow-\omega_\uparrow)t}\cdot i\underbrace{\langle\uparrow|\hat{S}_y|\downarrow\rangle}_{-i\frac{\hbar}{2}} - \tfrac{\sqrt{2}}{3}e^{+i(\omega_\downarrow-\omega_\uparrow)t}\cdot i\underbrace{\langle\downarrow|\hat{S}_y|\uparrow\rangle}_{+i\frac{\hbar}{2}}$$

$$= \tfrac{\sqrt{2}}{3}e^{-i(\omega_\downarrow-\omega_\uparrow)t}\cdot\tfrac{\hbar}{2} + \tfrac{\sqrt{2}}{3}e^{+i(\omega_\downarrow-\omega_\uparrow)t}\cdot\tfrac{\hbar}{2}$$

$$= \tfrac{\sqrt{2}}{3}\hbar\cos(\omega_\downarrow-\omega_\uparrow)t$$

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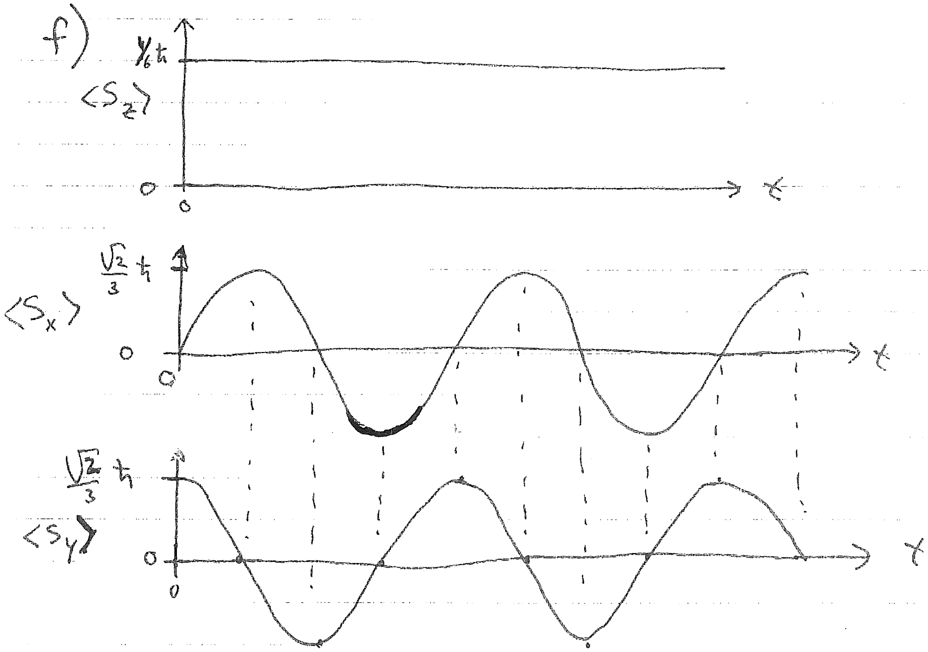
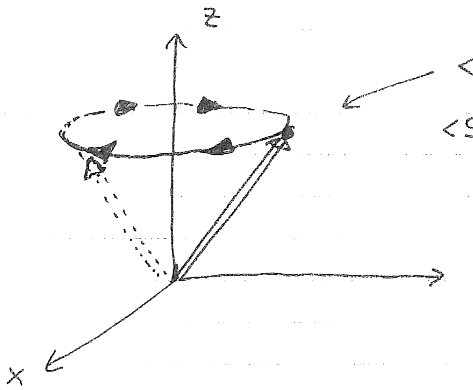
f)



The spin is precessing around the field $\vec{B} = B_z \cdot \vec{e}_z$ — unit vector in z-direction

See also Griffiths book p. 181



g) Each spin has a magnetic moment $\vec{\mu} = \gamma \vec{S}$. So if the spins precess all together there is a rotating macroscopic magnetic moment (magnetization). You can measure that with a coil (electrical wire in loops). The oscillating magnetization induces a current (or electromagnetic force e.m.f) in the coil that you can detect (like the dynamo on your bike for getting electrical current in your bike lights).

(5)

(1)

Problem 2

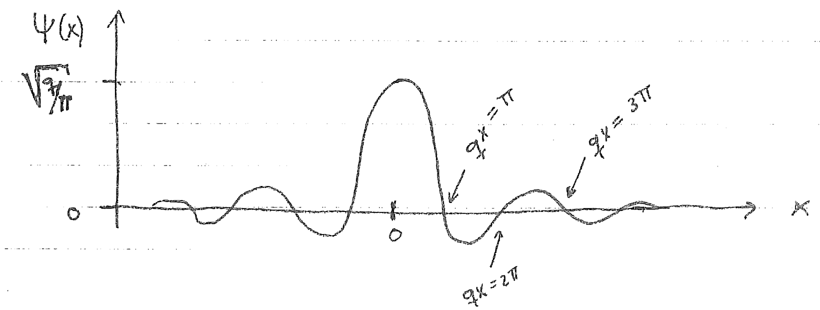
a)

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\Psi}(k) e^{ikx} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-q}^{q} \frac{1}{\sqrt{2q}} e^{ikx} dk = \frac{1}{\sqrt{4\pi q}} \left[\frac{1}{ix} e^{ikx} \right]_{-q}^{q}$$

$$= \frac{1}{\sqrt{4\pi q}} \left(\frac{1}{ix} e^{iqx} - \frac{1}{ix} e^{-iqx} \right) = \frac{1}{\sqrt{4\pi q}} \frac{2}{ix} i \sin(qx)$$

$$= \sqrt{\frac{q}{\pi}} \frac{\sin(qx)}{qx} = \sqrt{\frac{q}{\pi}} \operatorname{sinc}(qx)$$



b) $v_1 = 8000 \text{ m/s} \Rightarrow p_1 = m \cdot v_1 \Rightarrow k_1 = \frac{m v_1}{\hbar} = 0.8 \cdot 10^{10} \text{ m}^{-1}$

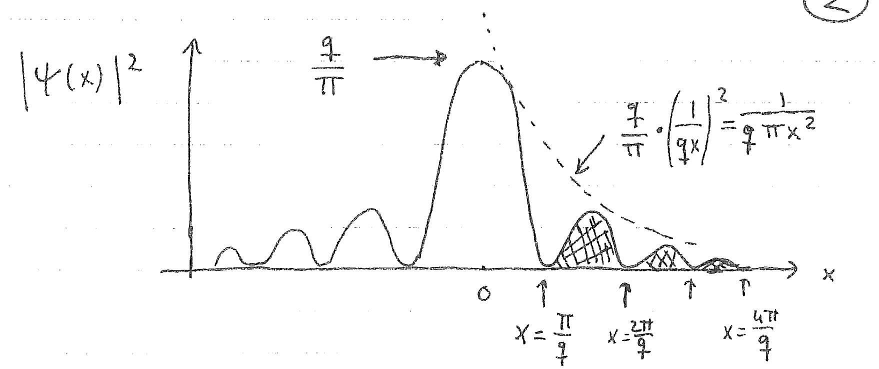
$v_2 = 10\,000 \text{ m/s} \Rightarrow p_2 = m v_2 \Rightarrow k_2 = \frac{m v_2}{\hbar} = 1.0 \cdot 10^{10} \text{ m}^{-1}$

$$P_{(v_1 < v < v_2)} = \int_{k_1}^{k_2} |\bar{\Psi}(k)|^2 dk = \int_{k_1}^{k_2} \frac{1}{2q} dk = \frac{k_2 - k_1}{2q}$$

$$= \frac{1}{2}(1 - 0.8) = 0.1$$

(2)

c)



$x_1 = 0.314$ nm $\Rightarrow$ in the plot at $x = \frac{\pi}{q}$

$x_2 = 4 \times 0.314$ nm $\Rightarrow$ in the plot at $x = \frac{4\pi}{q}$

So, the probability is the ⋙ area of the graph.

The main peak at $x=0$ has height $\frac{q}{\pi}$ (width is $\frac{2\pi}{q}$)

The peak at $x = \frac{3}{2}\frac{\pi}{q}$ has height $\frac{1}{q\pi \frac{9}{4}\frac{\pi^2}{q^2}} = \frac{q}{\pi}\cdot\frac{4}{9\pi^2} \approx 0.04\,\frac{q}{\pi}$
↳ width π/q

The peak at $x = \frac{5}{2}\frac{\pi}{q}$ has height $\frac{q}{\pi}\cdot\frac{4}{25\pi^2} \approx 0.016\cdot\frac{q}{\pi}$
(ignore next small peak)

Area of these 2 peaks on the side

$\approx \frac{1}{2} * (0.04 + 0.016)$ fraction of the main peak

Dominates the probability

$\Rightarrow$ area ≈ 1

$\Rightarrow$ Probability $\approx 2\%$

(3)

d) After the measurement of x, $\Delta x \approx 0.001$ nm

$\Rightarrow \Delta p_x \approx \frac{\hbar}{2\Delta x} \Rightarrow \Delta k \approx \frac{1}{2\Delta x} \approx 5\cdot 10^{11}\ m^{-1}$

$\Rightarrow \Delta k$ is about 50 times higher than for $\bar{\Psi}_0(k)$.

$\Rightarrow$ The value of $|\Psi(k)|^2$ is about 50 times lower.

So, the probability is about 50 times less than for the answer on question b).

e) $\bar{\Psi}(k,t) = e^{-\frac{i}{\hbar}\hat{H}t}\,\bar{\Psi}_0(k)$

$= e^{+\frac{i}{\hbar}\frac{\hbar^2 k^2}{2m}t}\,\bar{\Psi}_0(k) \Rightarrow$

$\hat{H} = -\frac{\hbar^2\hat{k}^2}{2m}$

$\Rightarrow$ in k-representation simply

$\hat{H} \leftrightarrow -\frac{\hbar^2 k^2}{2m}$

$$\bar{\Psi}(k,t) = \begin{cases} e^{+i\hbar\frac{k^2 t}{2m}}\cdot\frac{1}{\sqrt{2q}} & \text{, for } |k| < q \\ 0 & \text{, for all other } k \end{cases}$$

f) Besides the phase factors $e^{+i\hbar\frac{k^2 t}{2m}}$, the probability amplitudes for $\bar{\Psi}(k)$ are not changing in time. So, the probabilities to measure v between v_1 and v_2 do not change as a function of time

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g) The wave packet is spreading out in space (becoming wider) both in the +x direction and the −x direction. So, the probability to find it in the interval between x_1 and x_2 goes down. The wave packet gets wider at a speed that is about $v = \frac{\hbar q}{m}$.

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Problem 3

a) $|\vec{J}_1| = \sqrt{\frac{63}{4}}\,\hbar = \sqrt{j_1(j_1+1)}\,\hbar \Rightarrow j_1 = \frac{7}{2}$

This is a half-integer value, so atom 1 is a fermion.

$|\vec{J}_2| = \sqrt{2}\,\hbar = \sqrt{j_2(j_2+1)}\,\hbar \Rightarrow j_2 = 1$

This is an integer value, so atom 2 is a boson.

b) $\vec{J}_{tot} = \vec{J}_1 + \vec{J}_2 \Rightarrow$ follow rules for addition of angular momentum.

From a) we have $j_1 = \frac{7}{2}$, $j_2 = 1$. $|\vec{J}_{tot}|$ obeys

$$|\vec{J}_{tot}|\,|j_{tot}, m_{tot}\rangle = \sqrt{j_{tot}(j_{tot}+1)}\,\hbar\,|j_{tot}, m_{tot}\rangle$$

with $j_{tot} = |j_1 - j_2|,\ |j_1-j_2|+1,\ \dots\ |j_1+j_2|-1,\ |j_1+j_2|$

$\Rightarrow j_{tot} = \frac{5}{2}, \frac{7}{2}, \frac{9}{2} \Rightarrow$

Possible measurement outcomes are

$$\begin{cases} |\vec{J}_{tot}| = \sqrt{\frac{5}{2}\left(\frac{7}{2}\right)}\,\hbar = \sqrt{\frac{35}{4}}\,\hbar \\ |\vec{J}_{tot}| = \sqrt{\frac{7}{2}\cdot\left(\frac{9}{2}\right)}\,\hbar = \sqrt{\frac{63}{4}}\,\hbar \\ |J_{tot}| = \sqrt{\frac{9}{2}\cdot\left(\frac{11}{2}\right)}\,\hbar = \sqrt{\frac{99}{4}}\,\hbar \end{cases}$$

c) The possible measurement outcomes for $J_{z,tot}$ and $J_{x,tot}$ are the same if the starting point is that only $|\vec{J}_{tot}|$ is well defined. (2)

The possible outcomes for J_z obey

$$J_{z_{tot}} | j_{tot}, m_{tot} \rangle = m_{tot} \hbar \, | j_{tot}, m_{tot} \rangle$$

with $m_{tot} = -j_{tot}, -(j_{tot}-1) \; \cdots \; +(j_{tot}-1), +j_{tot} \Rightarrow$

Here all outcomes are covered by taking m_{tot} as

$$m_{tot} = -\frac{9}{2}, -\frac{7}{2}, -\frac{5}{2}, \ldots\ldots \frac{7}{2}, \frac{9}{2} \Rightarrow$$

The possible outcomes for $J_{x,tot}$ and $J_{z,tot}$ are

$$-\frac{9}{2}\hbar, \; -\frac{7}{2}\hbar, \; -\frac{5}{2}\hbar \; \ldots\ldots \; +\frac{7}{2}\hbar, \; +\frac{9}{2}\hbar \, .$$